An invitation to homological mirror symmetry

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(Simons Collaboration on Homological Mirror Symmetry)

Jacobi theta functions and counting triangles

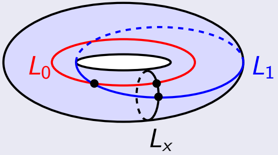
Jacobi theta function on the elliptic curve $E = \mathbb{C} / \mathbb{Z}+\tau\mathbb{Z}$

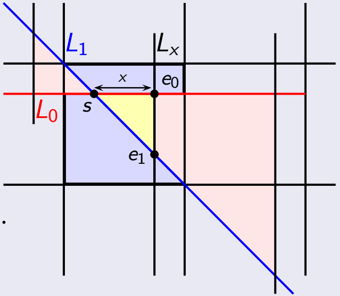
All doubly periodic holomorphic functions are constant, but we can ask for *quasi-periodic* functions: $s(z+1) = s(z)$, $s(z+\tau) = e^{-\pi i\tau - 2\pi i z} s(z)$

Only one up to scaling! $\quad s(z) = \vartheta(\tau; z) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n z)$.

(Jacobi, 1820s)

Counting triangles in $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ (weighted by area)





$$\boxed{?} = \cdots + T^{(x-1)^2/2} + T^{x^2/2} + T^{(x+1)^2/2} + \dots$$

$$= T^{x^2/2} \sum_{n\in\mathbb{Z}} T^{\frac{1}{2}n^2 + nx} = e^{\pi i \tau x^2} \vartheta(\tau; \tau x).$$

$(T = e^{2\pi i \tau})$

Homological mirror symmetry (Kontsevich 1994)

Algebraic (or analytic) geometry

Coherent sheaves (eg: $\mathcal{O}_V$, vector bundles $\mathcal{E} \to V$, skyscrapers $\mathcal{O}_{p\in V}$, ...)
Morphisms (+ extensions): $H^* hom(\mathcal{E},\mathcal{F}) = Ext^*(\mathcal{E},\mathcal{F})$.

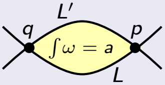
Derived category = complexes $0 \to \cdots \to \mathcal{E}^i \xrightarrow{d^i} \mathcal{E}^{i+1} \to \cdots \to 0 \;/\sim$
Eg: functions, intersections, cohomology...

$\Updownarrow$ **Mirror symmetry:** $D^b Coh(V) \simeq D^\pi \mathcal{F}(X,\omega)$

Symplectic geometry: **Fukaya category** $\mathcal{F}(X,\omega)$

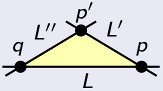
(X,ω) loc.$\simeq$ $(\mathbb{R}^{2n}, \sum dx_i \wedge dy_i)$, Lagrangian submanifolds L (*dim.* n, $\omega_{|L} = 0$).
Intersections (mod. Hamiltonian isotopy) = Floer cohomology

$CF^*(L,L') = \mathbb{K}^{|L\cap L'|}$
($\otimes$ local coefficients)



$\partial p = T^a q$

Product $CF(L',L'') \otimes CF(L,L') \to CF(L,L'')$: $\quad p' \cdot p = T^a q$

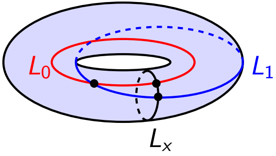


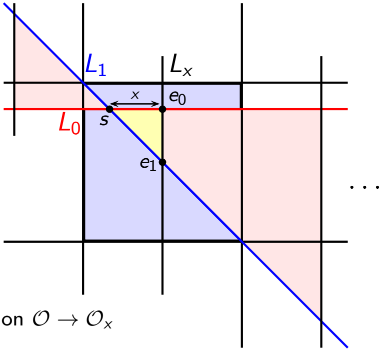
Example: elliptic curve (Polishchuk-Zaslow)

$$E = \mathbb{C}\,/\,\mathbb{Z}+\tau\mathbb{Z}, \quad \mathcal{L} = \mathbb{C}^2/(z,v) \sim (z+1,v) \sim (z+\tau, e^{-\pi i\tau - 2\pi i z}v)$$

$$\dim H^0(E,\mathcal{L}) = 1, \quad s(z) = \vartheta(\tau;z) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau + 2\pi i n z).$$

$$X = T^2 = \mathbb{R}^2/\mathbb{Z}^2$$





$$L_0 \xrightarrow{s} L_1 \xrightarrow{e_1} L_x$$

$e_1 \cdot s = \boxed{?}\; e_0$ $\qquad$ $e_0 \sim$ evaluation $\mathcal{O} \to \mathcal{O}_x$

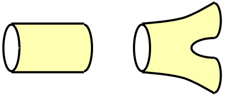
$$\boxed{?} = \dots + T^{(x-1)^2/2} + T^{x^2/2} + T^{(x+1)^2/2} + \dots$$

$$= T^{x^2/2} \sum_{n\in\mathbb{Z}} T^{\frac{1}{2}n^2+nx} = e^{\pi i\tau x^2}\vartheta(\tau;\tau x) \qquad (T = e^{2\pi i\tau})$$

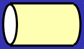
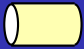
Homological mirror symmetry: towards a general setting

1. Projective Calabi-Yau varieties ($c_1 = 0$):
   T^2 (Polishchuk-Zaslow), T^{2n} (Kontsevich-Soibelman, Fukaya, Abouzaid-Smith), K3 surfaces (Seidel, Sheridan-Smith), $X_{d=n+2} \subset \mathbb{CP}^{n+1}$ (Sheridan), ...
2. Fano case: $\mathbb{CP}^n$, del Pezzo, toric varieties ... (LG models)
   (Kontsevich, Seidel, Auroux-Katzarkov-Orlov, Abouzaid, FOOO ...)
3. General type case, affine varieties, etc.
   Riemann surfaces, compact (Seidel, Efimov) or non-compact (Abouzaid-Auroux-Efimov-Katzarkov-Orlov, Lee, ...)
   hypersurfaces $\subset (\mathbb{C}^*)^n$ or toric varieties (Gammage-Shende, Abouzaid-Auroux, ...)
   ... and beyond

Goal of talk: give a flavor of this program
$\rightsquigarrow$ HMS for all Riemann surfaces starting with



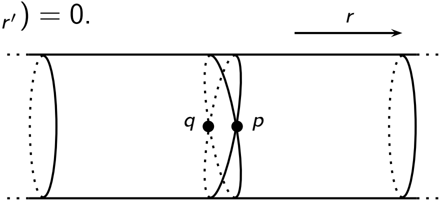
(focusing on HMS itself, ignoring developments from Strominger-Yau-Zaslow, skeleta, family Floer theory, etc.)

Example 1: $\mathcal{F}_c($  $) \simeq D^b_c($  $)$ (classical)

$X = \mathbb{R} \times S^1$, $\omega = dr \wedge d\theta$, $\quad L_r = \{r\} \times S^1$ (+ local system ξ)

$\Rightarrow HF^*(L_r, L_r) \simeq H^*(S^1, \mathbb{K})$,

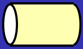
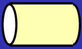
$\quad HF^*(L_r, L_{r'}) = 0.$



$\partial p = q - q = 0$

- $\mathcal{M}_{pt} = \{(L_r, \xi) \in \mathcal{F}(X)\}/\sim$ has a natural analytic structure
  Coordinate: $z(L_r, \xi) = T^r \mathrm{hol}(\xi) \in \mathbb{K}^*$.
  ($\forall L'$, $CF((L_r, \xi), L')$ has analytic dependence on z)
- $(L_r, \xi) \in \mathcal{F}(X, \omega) \longleftrightarrow \mathcal{O}_z \in D^b(X^\vee = \mathbb{K}^*)$

Strominger-Yau-Zaslow: X CY, $\pi : X \to B$ Lagrangian torus fibration
$\Rightarrow$ mirror $X^\vee = \{\mathcal{O}_p,\ p \in X^\vee\} = \big\{(L_b = \pi^{-1}(b), \xi) \in \mathcal{F}(X)\big\}/\sim$

Example 1: $\mathcal{F}_{wr}($$) \simeq D^b($$)$

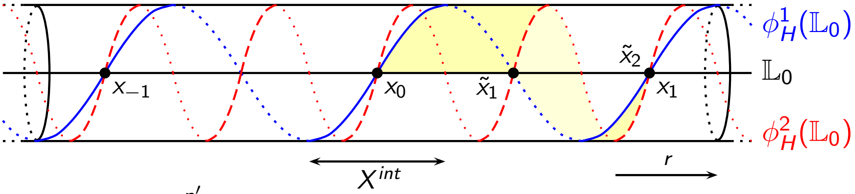
Abouzaid-Seidel
"wrapped Fukaya category"

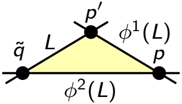
$X = \mathbb{R} \times S^1 \supset \mathbb{L}_0 = \mathbb{R} \times \{0\}$ non-compact Lagrangian.

Hamiltonian perturbation: $H = \frac{1}{2} r^2$, $\quad \phi_H^1(r,\theta) = (r, \theta + r)$.

($\to$ intersections $\in X^{int}$ + Reeb flow at boundary).

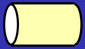
$$CW^*(\mathbb{L}_0, \mathbb{L}_0) := CF^*(\phi_H^1(\mathbb{L}_0), \mathbb{L}_0) = \bigoplus_{i \in \mathbb{Z}} \mathbb{K}\, x_i.$$



Product: 

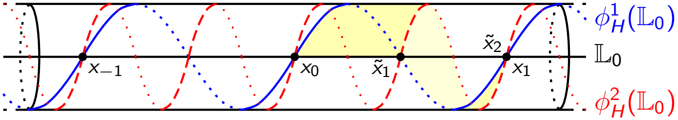
($\tilde{q} \in \phi^2(L) \cap L \leftrightarrow q \in \phi^1(L) \cap L$ via $r \mapsto 2r$)

$\boxed{x_k \cdot x_l = x_{k+l}} \Rightarrow \mathrm{End}(\mathbb{L}_0) \simeq \mathbb{K}[x^{\pm 1}]$. $\quad (x_k \rightsquigarrow x^k)$

Example 1: $\mathcal{F}_{wr}($  $) \simeq D^b(\mathbb{K}^*)$

Abouzaid-Seidel
"wrapped Fukaya category"

$X = \mathbb{R} \times S^1 \supset \mathbb{L}_0 = \mathbb{R} \times \{0\} \Rightarrow \mathrm{End}(\mathbb{L}_0) \simeq \mathbb{K}[x^{\pm 1}] \simeq \mathrm{End}(\mathcal{O}_{X^\vee})$.

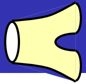


$\mathbb{L}_0$ generates $\mathcal{F}_{wr}(X)$.
Yoneda: $L \mapsto \mathrm{Hom}(\mathbb{L}_0, L)$ gives an embedding $\mathcal{F}_{wr}(X) \hookrightarrow \mathrm{End}(\mathbb{L}_0)$-mod.

Example: $(L_r, \xi) \mapsto HF(\mathbb{L}_0, (L_r, \xi)) \simeq \mathbb{K}[x^{\pm 1}]/(x - z) \quad (z = T^r \mathrm{hol}(\xi))$

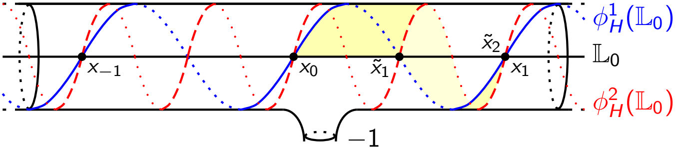
Theorem

$$\mathcal{F}_{wr}(X) \simeq \mathbb{K}[x^{\pm 1}]\text{-}mod \simeq D^b Coh(X^\vee).$$

Example 2: $\mathcal{F}_{wr}($  $)$ (Abouzaid-A.-Efimov-Katzarkov-Orlov)

$X = S^2 \setminus \{-1, 0, \infty\} = \mathbb{C}^* \setminus \{-1\}$, $\mathbb{L}_0 = \mathbb{R}_+$

$$\Rightarrow\ CW(\mathbb{L}_0, \mathbb{L}_0) = \bigoplus_{i\in\mathbb{Z}} \mathbb{K}\, x_i.$$

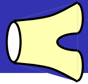


$$x_j \cdot x_i = \begin{cases} x_{i+j} & \text{if } ij \geq 0 \\ 0 & \text{if } ij < 0 \end{cases}$$

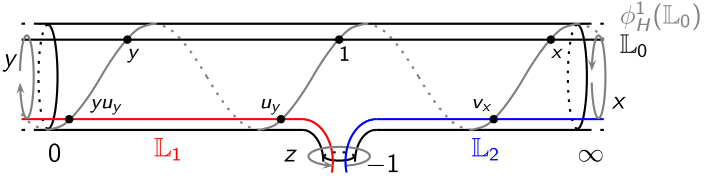
$$\Rightarrow\ \boxed{\mathrm{End}(\mathbb{L}_0) \simeq \mathbb{K}[x,y]/(xy=0).}$$

... $X^\vee = \mathrm{Spec}\,\mathbb{K}[x,y]/(xy=0) = \{xy=0\} \subset \mathbb{A}^2$?

$\mathcal{F}_{wr}(X) \hookrightarrow \mathrm{End}(\mathbb{L}_0)\text{-mod}$??

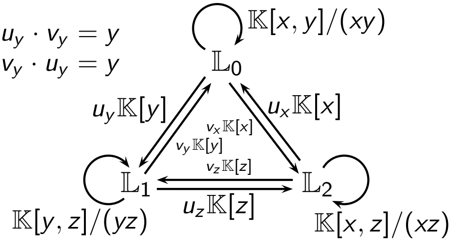
Example 2: $\mathcal{F}_{wr}($  $) \simeq D^b(\{xy=0\})$ (A-A-E-K-O)

$X = \mathbb{C}^* \setminus \{-1\}$: $\mathbb{L}_0 = (0,\infty)$, $\mathbb{L}_1 = (-1,0)$, $\mathbb{L}_2 = (-\infty,-1)$ generate



$u_y \cdot v_y = y$

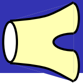
$v_y \cdot u_y = y$



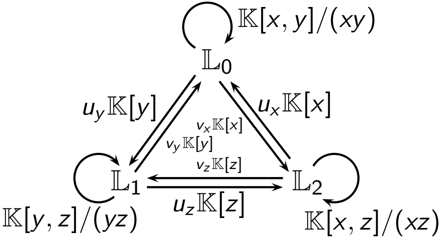
+ exact triangles

$$\mathbb{L}_2 \xrightarrow{u_x} \mathbb{L}_0 \xrightarrow{u_y} \mathbb{L}_1 \xrightarrow{u_z} \mathbb{L}_2[1]$$

$$\mathbb{L}_1 \xrightarrow{v_y} \mathbb{L}_0 \xrightarrow{v_x} \mathbb{L}_2 \xrightarrow{v_z} \mathbb{L}_1[1]$$

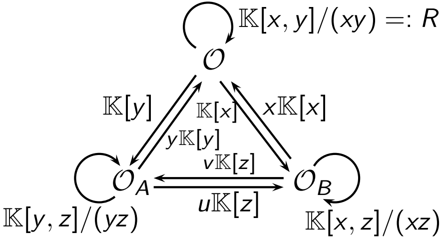
Example 2: $\mathcal{F}_{wr}($  $) \simeq D^b(\{xy=0\})$ (A-A-E-K-O)

$X = \mathbb{C}^* \setminus \{-1\} \supset \mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2$



$$\mathbb{L}_2 \xrightarrow{u_x} \mathbb{L}_0 \xrightarrow{u_y} \mathbb{L}_1 \xrightarrow{u_z} \mathbb{L}_2[1]$$
$$\mathbb{L}_1 \xrightarrow{v_y} \mathbb{L}_0 \xrightarrow{v_x} \mathbb{L}_2 \xrightarrow{v_z} \mathbb{L}_1[1]$$

$X^\vee = \{xy=0\} = A \cup B \subset \mathbb{A}^2$

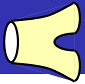


$\mathrm{Hom}(\mathcal{O}_A, \mathcal{O}_A) \simeq \mathbb{K}[y]$, $\mathrm{Ext}^{2k}(\mathcal{O}_A, \mathcal{O}_A) \ni z^k$.

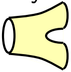
$$\mathcal{O}_B \xrightarrow{x} \mathcal{O} \xrightarrow{1} \mathcal{O}_A \xrightarrow{u} \mathcal{O}_B[1]$$
$$\mathcal{O}_A \xrightarrow{y} \mathcal{O} \xrightarrow{1} \mathcal{O}_B \xrightarrow{v} \mathcal{O}_A[1]$$

$\Rightarrow$ **Theorem (A-A-E-K-O)**

$$\mathcal{F}_{wr}(X) \simeq D^b Coh(X^\vee)$$

Example 2: $\mathcal{F}_{wr}($  $) \simeq D^b_{sing}(\mathbb{C}^3, -xyz)$ (A-A-E-K-O)

$X = \mathbb{P}^1 \setminus \{-1, 0, \infty\} \longleftrightarrow X^\vee = \{xy = 0\}$:

- $\mathcal{F}_{wr}(X) \simeq D^b Coh(\{xy = 0\})$ lacks symmetry in x, y, z.
- how to extend to higher genus? – gluing  ?

Stabilization: $X \simeq \{x + y + 1 = 0\} \subset (\mathbb{C}^*)^2$.

$$(\mathbb{X} = Bl((\mathbb{C}^*)^2 \times \mathbb{C}, X \times 0),\ W = p_{\mathbb{C}}) \longleftrightarrow (\mathbb{X}^\vee = \mathbb{C}^3,\ W^\vee = -xyz).$$

Theorem (A-A-E-K-O)

$\mathcal{F}_{wr}(X) \simeq D^b_{sing}(\mathbb{X}^\vee, W^\vee) := D^b Coh(\{xyz = 0\})/Perf$. *(Orlov)*

$(\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2) \longleftrightarrow ([\mathcal{O}_{\{z=0\}}], [\mathcal{O}_{\{x=0\}}], [\mathcal{O}_{\{y=0\}}])$

This result extends to all Riemann surfaces (AAEKO, Seidel, Efimov, H. Lee).
Mirror $(\mathbb{X}^\vee, W^\vee)$, $\dim \mathbb{X}^\vee = 3$. (Hori-Vafa, A-A-K)

Geometry of $(\mathbb{X}^\vee, W^\vee)$ (Hori-Vafa, Clarke, Abouzaid-A-Katzarkov, ...)

For an affine plane curve $\Sigma = \{f(x,y)=0\} \subset (\mathbb{C}^*)^2$, mirror:
$\mathbb{X}^\vee =$ toric CY 3-fold determined by *tropicalization* of f,
$W^\vee \in \mathcal{O}(\mathbb{X}^\vee)$, $Z := \{W^\vee = 0\} = \bigcup$ toric strata.
$\mathrm{sing}(Z) = \mathrm{crit}(W^\vee) = \bigcup$ 1-dim. strata $=$ union of $\mathbb{P}^1$ and $\mathbb{A}^1$.

Mirror decompositions: $\Sigma = \bigcup$  $\longleftrightarrow (\mathbb{X}^\vee, W^\vee) = \bigcup (\mathbb{C}^3, -xyz)$

Jeff Koons, *Balloon Dog* (photo Librado Romero - The New York Times)

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Mirror decompositions: $\Sigma = \bigcup$  $\longleftrightarrow (\mathbb{X}^\vee, W^\vee) = \bigcup (\mathbb{C}^3, -xyz)$

Theorem (Heather Lee)

$\mathcal{F}_{wr}(\Sigma) \simeq \lim \Big\{ \mathcal{F}_{wr}\big(\bigsqcup$  $\big) \rightrightarrows \mathcal{F}_{wr}\big(\bigsqcup$  $\big) \Big\} \simeq D^b_{sing}(\mathbb{X}^\vee, W^\vee)$ $(= D^b(Z)/Perf)$

(Related work: Bocklandt, Gammage-Shende, Lekili-Polishchuk, ...)

Theorem (Abouzaid-A.)

The converse also holds! $\mathcal{F}(\mathbb{X}^\vee, W^\vee) \simeq D^b Coh(\Sigma)$

(A.-Efimov-Katzarkov in progress recasts the l.h.s. in terms of $\mathrm{crit}(W^\vee) = \bigcup$ 1-d strata)
(see also C. Cannizzo's thesis for curves in abelian surfaces)
(Abouzaid-A. also holds for X = hypersurface or c.i. in $(\mathbb{C}^*)^n$)