Geometric Recursion

by

Jørgen Ellegaard Andersen
Center for Quantum Geometry of Moduli Spaces
Aarhus University
Denmark

at the

Inaugural Conference of the

Institute of the mathematical Sciences of the Americas

September 8, 2019.

Most of the work presented is joint with Gaëtan Borot and Nicolas Orantin.

Consider the following setting:

- $\mathcal{S}$ = Category of compact oriented surfaces (Morphisms are isotopy classes of diffeo's).
- $\mathcal{V}$ = Category of vector spaces.
- A functor $\mathbf{E} : \mathcal{S} \to \mathcal{V}$
- A functorial assigment $\Omega_\Sigma \in \mathbf{E}(\Sigma)$ for every object Σ of $\mathcal{S}$.

We note that in fact

$$\Omega_\Sigma \in \mathbf{E}(\Sigma)^{\Gamma_\Sigma},$$

where Γ_Σ is the mapping class group of Σ.

Consider the following setting:

- $\mathcal{S}$ = Category of compact oriented surfaces (Morphisms are isotopy classes of diffeo's).
- $\mathcal{V}$ = Category of vector spaces.
- A functor $\mathbf{E} : \mathcal{S} \to \mathcal{V}$
- A functorial assigment $\quad \Omega_\Sigma \in \mathbf{E}(\Sigma) \quad$ for every object Σ of $\mathcal{S}$.

We note that in fact

$$\Omega_\Sigma \in \mathbf{E}(\Sigma)^{\Gamma_\Sigma},$$

where Γ_Σ is the mapping class group of Σ.

Many construction in low dim. geometry and topology fit in this framework:

Consider the following setting:

- $\mathcal{S}$ = Category of compact oriented surfaces (Morphisms are isotopy classes of diffeo's).
- $\mathcal{V}$ = Category of vector spaces.
- A functor $\mathbf{E} : \mathcal{S} \to \mathcal{V}$
- A functorial assigment $\qquad \Omega_\Sigma \in \mathbf{E}(\Sigma) \qquad$ for every object Σ of $\mathcal{S}$.

We note that in fact

$$\Omega_\Sigma \in \mathbf{E}(\Sigma)^{\Gamma_\Sigma},$$

where Γ_Σ is the mapping class group of Σ.

Many construction in low dim. geometry and topology fit in this framework:

Ex. 1. The constant function one on Teichmüller space $\mathcal{T}_\Sigma$:

$$\mathbf{E}(\Sigma) = \mathcal{C}^0(\mathcal{T}_\Sigma), \qquad \Omega_\Sigma = 1 \in E(\Sigma)^{\Gamma_\Sigma}$$

Ex. 2. Sums over all simple closed multi-curves as a functions on Teichmüller space:

$$\mathbf{E}(\Sigma) = \mathcal{C}^0(\mathcal{T}_\Sigma), \qquad \Omega_\Sigma(\sigma) = \sum_{\gamma \in S_\Sigma} \prod_{c \in \pi_0(\gamma)} f(l_\sigma(\gamma_c)), \quad \sigma \in \mathcal{T}_\Sigma.$$

• S_Σ = **multi-curves** = the set of isotopy classes of embedded closed 1-dim. manifolds in Σ, such that no component is isotopic to a boundary component, nor are any two different components isotopic.
• $f : \mathbb{R}_+ \to \mathbb{C}$ is decaying sufficiently fast at infinity.

Ex. 3. Functions on Teichmüller space via spectral theory:

$$\mathbf{E}(\Sigma) = \mathcal{C}^0(\mathcal{T}_\Sigma), \qquad \Omega_\Sigma(\sigma) = \mathrm{Tr}(f(-\Delta_\sigma))$$

• $f : \mathbb{R} \to \mathbb{C}$ is sufficiently fast decaying at infinity and Δ_σ Dirichlet-Laplace-Beltrami operator on the Riemann surface Σ_σ, $\sigma \in \mathcal{T}_\Sigma$.

Ex. 4. Weil-Petersson symplectic form on Teichmüller space:

$\mathbf{E}(\Sigma) = \Omega^2(\mathcal{T}_\Sigma), \quad \Omega_\Sigma = \omega_{\mathrm{WP}}.$

Ex. 5. Bers complex structure I_{Bers} on Teichmüller space:

$\mathbf{E}(\Sigma) = C^\infty(\mathcal{T}_\Sigma, \mathrm{End}(T\mathcal{T}_\Sigma)), \quad \Omega_\Sigma = I_{\mathrm{Bers}}.$

Ex. 6. Closed form on Teichmüller space:

$\mathbf{E}(\Sigma) = \Omega^*(\mathcal{T}_\Sigma), \quad \Omega_\Sigma \in \Omega^*(\mathcal{T}_\Sigma)^{\Gamma_\Sigma}, \; d\Omega_\Sigma = 0.$

• Representing non-trivial cohomology classes on moduli space of curves $\mathcal{M}(\Sigma) = \mathcal{T}_\Sigma/\Gamma_\Sigma$.

Ex. 7. Fock-Rosly Poisson structure P_{FR} on moduli spaces of flat connections $M_G(\Sigma)$:

$\mathbf{E}(\Sigma) = C^\infty(M_G(\Sigma), \Lambda^2 TM_G(\Sigma)), \quad \Omega_\Sigma = P_{\mathrm{FR}} \in E(\Sigma)^{\Gamma_\Sigma}.$

• G any semi-simple Lie group either complex or real.

Ex. 8. Narasimhan-Seshadri complex structure on moduli spaces of flat connections $M_G(\Sigma, c)$:

$\mathbf{E}(\Sigma) = C^\infty(\mathcal{T}_\Sigma, C^\infty(M_G(\Sigma, c), \mathrm{End}(TM_G(\Sigma, c))), \quad \Omega_\Sigma = I_{NS} \in E(\Sigma)^{\Gamma_\Sigma}.$

• G any real semi-simple Lie group and c is an assignment of conjugacy classes to each boundary components of Σ, in which we assume the holonomy around each boundary component is contained.

Ex. 9. Ricci potentials on the moduli spaces of flat connections $M_G(\Sigma, c)$:

$\mathbf{E}(\Sigma) = C^\infty(\mathcal{T}_\Sigma, C^\infty(M_G(\Sigma, c))), \quad \Omega_\Sigma = F_{\text{Ricci}} \in E(\Sigma)^{\Gamma_\Sigma}$.

Ex. 10. Hitchin's Hyper-Kähler structure on moduli spaces of parabolic Higgs bundles:

$\mathbf{E}(\Sigma) = C^\infty(\mathcal{T}_\Sigma, C^\infty(M_G(\Sigma, c), \text{End}(TM_G(\Sigma, c)))^{\times 3}, \quad \Omega_\Sigma = (I, J, K)_{\text{Hitchin}} \in E(\Sigma)^{\Gamma_\Sigma}$.

- G is a complex semi-simple Lie group and c is as before.

Ex. 11. Representations of mapping class groups $\rho : \Gamma_\Sigma \to \text{Aut}(V)$:

$\mathbf{E}(\Sigma) = \Omega^1(\mathcal{T}, \mathcal{T} \times \text{End}(V)), \quad \Omega_\Sigma = u_\rho \in E(\Sigma)^{\Gamma_\Sigma}$.

Ex. 12. Boundary vectors in TQFT Z:

$\mathbf{E}(\Sigma) = Z(\Sigma), \quad \Omega_\Sigma = Z(X^3) \in E(\Sigma)^{\Gamma_X}, \partial X = \Sigma$.

Ex. 13. Any invariant I_3 of closed oriented 3-manifolds:

$\mathbf{E}(\Sigma) = \mathbb{C}[\text{Heegaard diagrams } (\alpha, \beta) \text{ on } \Sigma]^*, \quad \Omega_\Sigma = I_3 \in E(\Sigma)^{\Gamma_\Sigma}, I_3(\alpha, \beta) = I_3(X^3_{(\alpha,\beta)})$.

Ex. 14. Any invariant I_4 of smooth closed oriented 4-manifolds:

$\mathbf{E}(\Sigma) = \mathbb{C}[\text{Tri-section diagrams } (\alpha, \beta, \gamma) \text{ on } \Sigma]^*, \Omega_\Sigma = I_4 \in E(\Sigma)^{\Gamma_\Sigma}, I_4(\alpha, \beta, \gamma) = I_4(X^4_{(\alpha,\beta,\gamma)})$.

Ex. 15. Closed forms representing cohomology classes from Gromov-Witten Theory:

$\mathbf{E}(\Sigma) = \Omega^*(\mathcal{T}_\Sigma), \quad \Omega_\Sigma = \varphi_{GW} \in E(\Sigma)^{\Gamma_\Sigma}$.

Ex. 16. Amplitudes in closed string theory:

$\mathbf{E}(\Sigma) = \Omega^{\text{top}}(\mathcal{T}_\Sigma), \quad \Omega_\Sigma = A_\Sigma \in E(\Sigma)^{\Gamma_\Sigma}$.

The category of surfaces we consider $\mathcal{S}$:

Objects: Compact oriented surfaces Σ of negative Euler characteristic with a marked point on each boundary component together with an orientation of the boundary, such that $\partial\Sigma = \partial_-\Sigma \cup \partial_+\Sigma$, and such that the inclusion map $\partial_-\Sigma \subset \Sigma$ induces $\pi_0(\partial_-\Sigma) \cong \pi_0(\Sigma)$.

Morphisms: Isotopy classes of orientation preserving diffeomorphisms which preserves marked points and orientations on the boundary modulo isotopies which also preserves all this structure.

The category of surfaces we consider $\mathcal{S}$:

Objects: Compact oriented surfaces Σ of negative Euler characteristic with a marked point on each boundary component together with an orientation of the boundary, such that $\partial\Sigma = \partial_-\Sigma \cup \partial_+\Sigma$, and such that the inclusion map $\partial_-\Sigma \subset \Sigma$ induces $\pi_0(\partial_-\Sigma) \cong \pi_0(\Sigma)$.

Morphisms: Isotopy classes of orientation preserving diffeomorphisms which preserves marked points and orientations on the boundary modulo isotopies which also preserves all this structure.

The category of vector spaces $\mathcal{V}$:

Objects: Hausdorff, complete, locally convex topological vector spaces over $\mathbb{C}$.

Morphisms: Morphisms of locally convex topological vector spaces.

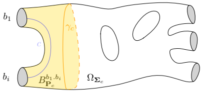
Suppose now we have a functor

$$\mathbf{E} : \mathcal{S} \to \mathcal{C}.$$

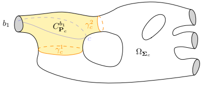
We want to recursively define for every object Σ of $\mathcal{S}$

$$\Omega_\Sigma \in \mathbf{E}(\Sigma)^{\Gamma_\Sigma}$$

recursing in the Euler characteristic $\chi = \chi(\Sigma)$. The basic idea is to recursively remove **pairs of pants** which are embedded around the components of $\partial_-\Sigma$, so that χ goes up by one in each step ending with $\chi = -1$ which is a pair of pants P or a one holed torus T.



The B case.



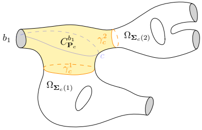
The C case.

This will require:

- Disjoint union morphisms: $\sqcup : \mathbf{E}(\Sigma_1) \times \mathbf{E}(\Sigma_2) \to \mathbf{E}(\Sigma_1 \sqcup \Sigma_2)$
- Glueing morphisms: $\Theta_\beta : \mathbf{E}(\Sigma_1) \times \mathbf{E}(\Sigma_2) \to \mathbf{E}(\Sigma_1 \cup_\beta \Sigma_2)$

for subset $\beta \subset \pi_0(\partial_+\Sigma_1) \times \pi_0(\partial_-\Sigma_2)$ consisting of disjoint pairs.

- Starting data $A \in \mathbf{E}(P)^{\Gamma_P}$, $D \in \mathbf{E}(T)^{\Gamma_T}$ giving $\Omega_P = A, \Omega_T = D$.
- Recursion data B^b $(b \in \pi_0(\partial_+ P))$, $C \in \mathbf{E}(P)$.



The C case.

But in order to have mapping class group invariance persist through the recursion, we will also need to be able to make sense of the following infinite sum

$$\Omega_\Sigma = \sum_{P \in \mathcal{P}_B(\Sigma)} \Theta_{b'}(B^b, \Omega_{\Sigma_c}) + \sum_{P \in \mathcal{P}_C(\Sigma)} \Theta_{b,b'}(C, \Omega_{\Sigma_c}).$$

where $\mathcal{P}_B(\Sigma)$ and $\mathcal{P}_C(\Sigma)$ are the sets of isotopy classes of embeddings of pair of pants into Σ of type B and C respectively and $\partial_+ P = b \cup b'$.

Definition

Initial data for a given target theory **E** are assignments

- $A, C \in \mathbf{E}(P)^{\Gamma_P}$.
- $B^b \in \mathbf{E}(P)$ for $b \in \pi_0(\partial_+ P)$ such that $\varphi(B^b) = B^{\varphi(b)}$ for all $\varphi \in \Gamma(P)$.
- $D \in \mathbf{E}(T)^{\Gamma_T}$.

Definition

Initial data for a given target theory **E** are assignments

- $A, C \in \mathbf{E}(P)^{\Gamma_P}$.
- $B^b \in \mathbf{E}(P)$ for $b \in \pi_0(\partial_+ P)$ such that $\varphi(B^b) = B^{\varphi(b)}$ for all $\varphi \in \Gamma(P)$.
- $D \in \mathbf{E}(T)^{\Gamma_T}$.

Definition

The initial data is called *admissible* if A, B, C, D satisfies certain decay properties.

Let (A, B, C, D) be an admissible initial data for a target theory $\mathbf{E}$.

Definition

$$\bullet\ \Omega_{\emptyset} := 1 \in E(\emptyset) = \mathbb{K}, \quad \bullet\ \Omega_P := A, \quad \bullet\ \Omega_T := D.$$

For Σ a connected object of $\mathcal{S}$ with Euler characteristic $\chi(\Sigma) \leq -2$ we seek to inductively define

$$\bullet \quad \Omega_{\Sigma} := \frac{1}{2} \sum_{P \in \mathcal{P}_C(\Sigma)} \Theta_{b,b'}(C, \Omega_{\Sigma_c}) + \sum_{P \in \mathcal{P}_B(\Sigma)} \Theta_{b'}(B^b_{\mathbf{P}_c}, \Omega_{\Sigma_c})$$

as an element of $\mathbf{E}(\Sigma)$.
For disconnected objects Σ, we declare

$$\Omega_{\Sigma} := \bigsqcup_{a \in \pi_0(\Sigma)} \Omega_{\Sigma(a)}.$$

Theorem (Andersen, Borot and Orantin)

The assignment $\Sigma \mapsto \Omega_{\Sigma}$ is well-defined. More precisely, the above series defining Ω_{Σ} converges absolutely for any of the seminorms of $\mathbf{E}(\Sigma)$, and it is functorial. In particular,

$$\Omega_{\Sigma} \in \mathbf{E}(\Sigma)^{\Gamma_{\Sigma}}.$$

- Recall that S_Σ is Thurston's set of multi curves on Σ.
- The basic idea is to consider functions

$$l : S_\Sigma \to \mathbb{R}_+$$

for which there exists $c_\Sigma, d_\Sigma \in \mathbb{R}_+$ such that

$$\#\{\gamma \in S_\Sigma | l(\gamma) < L\} \;\leq\; c_\Sigma L^{d_\Sigma} \quad \forall L \in \mathbb{R}_+.$$

- The sets of pair of pants $\mathcal{P}_B(\Sigma)$ are really just subsets of S_Σ and we see that

$$\zeta_B(s) = \sum_{P \in \mathcal{P}_B(\Sigma)} l(P)^{-s}$$

are well defined functions for $s > d_\Sigma + 1$.

- If we now assume that for each $P \in \mathcal{P}_B(\Sigma)$ we have the esimate

$$\left|\Theta_{b'}(B^b_{\mathbf{P}_c}, \Omega_{\Sigma_c})\right| \;\leq\; C\left|\Omega_{\Sigma_c}\right| l(P)^{-(d_\Sigma+2)}$$

then we get that

$$\sum_{P \in \mathcal{P}_B(\Sigma)} \left|\Theta_{b'}(B^b_{\mathbf{P}_c}, \Omega_{\Sigma_c})\right| \leq C\left|\Omega_{\Sigma_c}\right| \zeta_B(d_\Sigma + 2)\,.$$

Same argument of course works for $\mathcal{P}_C(\Sigma)$.

Topological Recursion

- Invented by Chekhov, Eynard, Orantin around 2005-07 and written down by Eynard and Orantin.
- Takes as its input a spectral curve together with a certain one form and two form on a two fold product of the spectral curve.
- It produces forms index by non-negative integers g and n on products of the spectral curve, which are defined by a recursion with a structure very reminiscent of the structure of the irreducible components of the boundary divisor of $M_{g,n}$'s.

Topological Recursion

- Invented by Chekhov, Eynard, Orantin around 2005-07 and written down by Eynard and Orantin.
- Takes as its input a spectral curve together with a certain one form and two form on a two fold product of the spectral curve.
- It produces forms index by non-negative integers g and n on products of the spectral curve, which are defined by a recursion with a structure very reminiscent of the structure of the irreducible components of the boundary divisor of $M_{g,n}$'s.

Quantum Airy structures

- Invented by Kontevich and Soibelman in 2016-17.
- Takes as input four (maybe infinite) tensors A, B, C, D which is the data needed to specify and quantize a certain quadratic Lagrangian.
- For any initial data for TR one can construct an A, B, C, D which gives a Quantum Airy structure and the output of TR becomes incoded in Kontsevich and Soibelmans general construction of the quantization of the quadratic Lagrangian.

Topological Recursion

- Invented by Chekhov, Eynard, Orantin around 2005-07 and written down by Eynard and Orantin.
- Takes as its input a spectral curve together with a certain one form and two form on a two fold product of the spectral curve.
- It produces forms index by non-negative integers g and n on products of the spectral curve, which are defined by a recursion with a structure very reminiscent of the structure of the irreducible components of the boundary divisor of $M_{g,n}$'s.

Quantum Airy structures

- Invented by Kontevich and Soibelman in 2016-17.
- Takes as input four (maybe infinite) tensors A, B, C, D which is the data needed to specify and quantize a certain quadratic Lagrangian.
- For any initial data for TR one can construct an A, B, C, D which gives a Quantum Airy structure and the output of TR becomes incoded in Kontsevich and Soibelmans general construction of the quantization of the quadratic Lagrangian.

Geometric Recursion

- We (Andersen, Borot, Orantin) invented it gradually during the period 2015-18.
- Our first version of Geometric Recursion was based on the spectral curve technology.
- The A, B, C, D formalism presented above was inspired by Kontsevich and Soibelmans reformulation of TR and simplified our constructions considerably.
- GR is rather different in the sense that it involves something functorially defined on surfaces which do have a genus g and a number of boundary components n.
- As we will see below, for certain target theories, GR can be mapped to TR and it is a means to establish that something can be computed by means of TR.

Let Σ be an object of $\mathcal{S}$, e.g. Σ is a pointed bordered surface, so we have marked points on the boundary $o = (o_b)_{b \in \pi_0(\partial\Sigma)}$.

Definition

The Teichmüller space $\mathcal{T}^p_\Sigma$ for a pointed bordered surface Σ is

$$\{\mu : \Sigma \to S \mid S \text{ bordered Riemann Surface}\}/\sim$$

Here $(\mu_1 : \Sigma \to S_1) \sim (\mu_2 : \Sigma \to S_2)$ iff there exist $\Phi : S_1 \to S_2$ biholomorphism s.t. $\mu_2^{-1} \circ \Phi \circ \mu_1$ restricts to the identity on o and is isotopic to Id_Σ via diffeomorphism which also restrict to the identity on o.

The canonical projection

$$p_\Sigma \ : \ \mathcal{T}^p_\Sigma \longrightarrow \mathcal{T}_\Sigma,$$

is an $\mathbb{R}^{\pi_0(\partial\Sigma)}$-bundle.
The group Δ_Σ generated by boundary parallel Dehn twist acts free on $\mathcal{T}^p_\Sigma$ and we denote $\widetilde{\mathcal{T}}^p_\Sigma := \mathcal{T}^p_\Sigma/\Delta_\Sigma$. Then the induced projection

$$\tilde{p}_\Sigma \ : \ \widetilde{\mathcal{T}}^p_\Sigma \longrightarrow \mathcal{T}_\Sigma$$

is a $U(1)^{\pi_0(\Sigma)}$-bundle.

For our pair of pants P, we get a canonical identification

$$\mathcal{T}_P \cong \mathbb{R}_+^3$$

and isomorphism

$$\mathcal{T}_P^p \cong (\mathbb{R}_+ \times \mathbb{R})^3, \qquad \widetilde{\mathcal{T}}_P^p \cong (\mathbb{R}_+ \times U(1))^3 .$$

We denote by $(L_i, \theta_i)_{i=1}^3$ the resulting coordinates on $\widetilde{\mathcal{T}}_{\mathbf{P}}^p$.

For our pair of pants P, we get a canonical identification

$$\mathcal{T}_P \cong \mathbb{R}_+^3$$

and isomorphism

$$\mathcal{T}_P^p \cong (\mathbb{R}_+ \times \mathbb{R})^3, \qquad \widetilde{\mathcal{T}}_P^p \cong (\mathbb{R}_+ \times U(1))^3 \,.$$

We denote by $(L_i, \theta_i)_{i=1}^3$ the resulting coordinates on $\widetilde{\mathcal{T}}_{\mathbf{P}}^p$.

For Σ_i objects of $\mathcal{S}$ and $\beta \subset \pi_0(\partial_+\Sigma_1) \times \pi_0(\partial_-\Sigma_2)$ we obtain by gluing a new object $\Sigma_1 \cup_\beta \Sigma_2$.
We have the following inclusion map

$$\iota_b : \widetilde{\mathcal{T}}_{\Sigma_1 \cup \Sigma_2}^{p,=_\beta} \to \widetilde{\mathcal{T}}_{\Sigma_1 \cup \Sigma_2}^p$$

where $\widetilde{\mathcal{T}}_{\Sigma_1 \cup \Sigma_2}^{p,=_\beta}$ is the subset of $\widetilde{\mathcal{T}}_{\Sigma_1 \cup \Sigma_2}^p$ where the length of the glued boundary components match. Then we have a $U(1)^{|\beta|}$-fibration

$$\widetilde{\vartheta}_\beta : \widetilde{\mathcal{T}}_{\Sigma_1 \cup \Sigma_2}^{p,=_\beta} / \Delta_\beta \to \widetilde{\mathcal{T}}_{\Sigma_1 \cup_\beta \Sigma_2}^p$$

obtained by glueing.
Here Δ_β is the group generated by pairs of opposite Dehn-twist along each boundary pair of β, which cancel after glueing.

Union morphisms

As union morphism, we take $\sqcup : \mathbf{E}(\Sigma_1) \times \mathbf{E}(\Sigma_2) \to \mathbf{E}(\Sigma_1 \cup \Sigma_2)$ given by $f_1 \sqcup f_2 = q_1^* f_1 \cdot q_2^* f_2$, where $q_i : \mathbf{E}(\Sigma_1 \cup \Sigma_2) \to \mathbf{E}(\Sigma_i)$ are the projections.

Glueing morphisms

For $(f_1, f_2) \in \mathbf{E}(\Sigma_1) \times \mathbf{E}(\Sigma_2)$ we define

$$\Theta_b(f_1, f_2)(\sigma) := \int_{\tilde{\vartheta}_\gamma^{-1}(\sigma)} \iota_b^*(f_1 \sqcup f_2)\, d\alpha.$$

where $d\alpha$ is the rotation invariant measure on the fibers of $\tilde{\vartheta}_\gamma$.

Initial data

- $A, C \in \mathcal{C}^0(\widetilde{\mathcal{T}}_P^p)^{\Gamma_P} \cong \mathcal{C}^0((\mathbb{R}_+ \times U(1))^{\times 3})^{S_2}$
- $B^b, B^{b'} \in \mathcal{C}^0(\widetilde{\mathcal{T}}_P^p) \cong \mathcal{C}^0((\mathbb{R}_+ \times U(1))^{\times 3})$ ($B^{b'}$ is B^b with last two coordinates permuted.)
- $D \in \mathcal{C}^0(\widetilde{\mathcal{T}}_T^p)^{\Gamma_T}$

Admissibility: For all $s, \varepsilon > 0$ there exist $M(s, \varepsilon)$ s.t.

$$\sup_{\sigma \in K_P(\varepsilon)} (1 + [l_\sigma(\partial_+ P) - l_\sigma(\partial_- P)]_+)^s \left|B^b(\sigma)\right| \leq M(s, \varepsilon)$$

$$\sup_{\sigma \in K_P(\varepsilon)} (1 + [l_\sigma(\partial_+ P) - l_\sigma(\partial_- P)]_+)^s \left|C(\sigma)\right| \leq M(s, \varepsilon)\,.$$

Here $K_\Sigma(\varepsilon) := \{\sigma \in \widetilde{\mathcal{T}}_\Sigma^p \mid \mathrm{sys}_\sigma \geq \varepsilon\}$ and $\left([x]_+ = \frac{1+\mathrm{Sign(x)}}{2} x\right)$

Consider the Mirzakhani-McShane initial data:

$$\begin{aligned} A_{\mathrm{MM}}(L_1, L_2, L_3) &= 1 \\ B_{\mathrm{MM}}(L_1, L_2, \ell) &= 1 - \frac{1}{L_1} \ln\left(\frac{\cosh\left(\frac{L_2}{2}\right) + \cosh\left(\frac{L_1+\ell}{2}\right)}{\cosh\left(\frac{L_2}{2}\right) + \cosh\left(\frac{L_1-\ell}{2}\right)}\right) \\ C_{\mathrm{MM}}(L_1, \ell, \ell') &= \frac{1}{L_1} \ln\left(\frac{\exp(\frac{L_1}{2}) + \exp(\frac{\ell+\ell'}{2})}{\exp(-\frac{L_1}{2}) + \exp(\frac{\ell+\ell'}{2})}\right) \end{aligned}$$

and

$$D_{\mathrm{MM}}(\sigma) = \sum_{\gamma \in S_T} C_{\mathrm{MM}}(\ell_\sigma(\partial T), \ell_\sigma(\gamma), \ell_\sigma(\gamma))$$

for $\sigma \in \widetilde{\mathcal{T}}^p_T$.

Theorem (Andersen, Borot and Orantin)

For any object Σ in $\mathcal{S}$ the Geometric Recursion applied to the initial data $A_{MM}, B_{MM}, C_{MM}, D_{MM}$ for the target theory $\mathbf{E}(\Sigma) = \mathcal{C}^0(\widetilde{\mathcal{T}}^p_\Sigma)$ gives

$$\Omega_\Sigma = 1.$$

Consider the Kontsevich initial data:

$$\begin{aligned} A_K(L_1, L_2, L_3) &= 1 \\ B_K(L_1, L_2, \ell) &= \frac{1}{2L_1}\left([L_1 - L_2 - \ell]_+ - [-L_1 + L_2 - \ell]_+ + [L_1 + L_2 - \ell]_+\right) \\ C_K(L_1, \ell, \ell') &= \frac{1}{L_1}[L_1 - \ell - \ell']_+ \end{aligned}$$

and

$$D_K(\sigma) = \sum_{\gamma \in S_T} C_K(\ell_\sigma(\partial T), \ell_\sigma(\gamma), \ell_\sigma(\gamma))$$

for $\sigma \in \widetilde{\mathcal{T}}_T^p$.

Theorem (Andersen, Borot and Orantin)

For any object Σ in $\mathcal{S}$ the Geometric Recursion applied to the initial data A_K, B_K, C_K, D_K for the target theory $\mathbf{E}(\Sigma) = \mathcal{C}^0(\widetilde{\mathcal{T}}_\Sigma^p)$ gives

$$\Omega_\Sigma^{\mathrm{K}} \in C^0(\mathcal{M}_\Sigma)$$

which is integrable over $\mathcal{M}_\Sigma(L_1, \ldots, L_n)$ w.r.t. $\nu_\Sigma(L_1, \ldots, L_n)$ and

$$\int_{\mathcal{M}_\Sigma(L_1,\ldots,L_n)} \Omega_\Sigma^{\mathrm{K}} \nu_\Sigma(L_1, \ldots, L_n) = \int_{\overline{\mathcal{M}}_{g,n}} \exp\left(\sum_{i=1}^n \frac{L_i^2}{2}\psi_i\right),$$

where

- *$\nu_\Sigma(L_1, \ldots, L_n)$ Weil-Petersson volume form on $\mathcal{M}_\Sigma(L_1, \ldots, L_n)$*
- *ψ_i are the Psi-classes of $\overline{\mathcal{M}}_{g,n}$ and g is the genus of Σ.*

Let $f : \mathbb{R}_+ \to \mathbb{C}$ be a continuous function. For any object Σ of $\mathcal{S}$ we consider the series

$$F_\Sigma(\sigma) = \sum_{\gamma \in S_\Sigma} \prod_{c \in \pi_0(\gamma)} f(\ell_\sigma(\gamma_c))$$

where S_Σ is the set multi-curves on Σ.

Let us denote

$$s_f := \inf \left\{ s \in \mathbb{R}_+ \quad \middle| \quad \forall \epsilon > 0, \quad \sup_{\ell \geq \epsilon} \ell^s |f(\ell)| < +\infty \right\}$$

If Σ is a connected bordered surface with genus g and n boundary components such that $6g - 6 + 2n < s_f$, then

$$F_\Sigma(\sigma) = \sum_{\gamma \in S_\Sigma} \prod_{c \in \pi_0(\gamma)} f(\ell_\sigma(\gamma_c))$$

is absolutely convergent and defines a continuous function of $\sigma \in \mathcal{T}_\Sigma$. Since Γ_Σ acts by permutations on S_Σ, this function is Γ_Σ-invariant
This function is obviously multiplicative for disjoint unions

$$F_{\Sigma_1 \sqcup \Sigma_2} = F_{\Sigma_1} F_{\Sigma_2}.$$

We observe that for a pair of pants P we have that $F_P = 1$.

f-twisted Mirzakhani-McShane initial data:

$$B^f_{\mathrm{MM}}(L_1, L_2, \ell) = B_{\mathrm{MM}}(L_1, L_2, \ell) + f(\ell)$$

$$C^f_{\mathrm{MM}}(L_1, \ell, \ell') = C_{\mathrm{MM}}(L_1, \ell, \ell') + B_{\mathrm{MM}}(L_1, \ell, \ell')f(\ell) + B_{\mathrm{MM}}(L_1, \ell', \ell)f(\ell') + f(\ell)f(\ell').$$

$$A^f_{\mathrm{MM}} = 1, \quad D^f_{\mathrm{MM}}(\sigma) = 1 + \sum_{\gamma \in S_T} f(\ell_\sigma(\gamma)),$$

Theorem (Andersen, Borot and Orantin)

For any object Σ in $\mathcal{S}$ the Geometric Recursion applied to the initial data $A^f_{MM}, B^f_{MM}, C^f_{MM}, D^f_{MM}$ for the target theory $\mathbf{E}(\Sigma) = \mathcal{C}^0(\widetilde{T}^p_\Sigma)$ gives

$$\Omega_\Sigma = F_\Sigma.$$

f-twisted Mirzakhani-McShane initial data:

$$B^f_{\mathrm{MM}}(L_1, L_2, \ell) = B_{\mathrm{MM}}(L_1, L_2, \ell) + f(\ell)$$

$$C^f_{\mathrm{MM}}(L_1, \ell, \ell') = C_{\mathrm{MM}}(L_1, \ell, \ell') + B_{\mathrm{MM}}(L_1, \ell, \ell')f(\ell) + B_{\mathrm{MM}}(L_1, \ell', \ell)f(\ell') + f(\ell)f(\ell').$$

$$A^f_{\mathrm{MM}} = 1, \quad D^f_{\mathrm{MM}}(\sigma) = 1 + \sum_{\gamma \in S_T} f(\ell_\sigma(\gamma)),$$

Theorem (Andersen, Borot and Orantin)

For any object Σ in $\mathcal{S}$ the Geometric Recursion applied to the initial data $A^f_{MM}, B^f_{MM}, C^f_{MM}, D^f_{MM}$ for the target theory $\mathbf{E}(\Sigma) = \mathcal{C}^0(\widetilde{T}^p_\Sigma)$ gives

$$\Omega_\Sigma = F_\Sigma.$$

Main idea of the proof is that for a given $\gamma \in S_\Sigma$, there **always exist a pair of pants** in Σ around $\partial_-\Sigma$, which does **not intersect** γ.

If $\Phi \in \mathcal{C}^0(\mathcal{T}_\Sigma)^{\Gamma_\Sigma}$ is integrable with respect to the Weil-Petersson volume form ν_Σ, we define the expectation value

$$\langle \Phi \rangle (L_1, \dots, L_n) = \int_{\mathcal{M}_{\Sigma_{g,n}}(L_1,\dots,L_n)} \Phi \, d\nu_\Sigma$$

Theorem (Andersen, Borot and Orantin)

$$\begin{aligned}
&\langle F_{\Sigma_{g,n}} \rangle (L_1, \dots, L_n) = \\
&\quad \sum_{m=2}^{n} \int_{\mathbb{R}_+} B^f(L_1, L_m, \ell) \, \langle F_{\Sigma_{g,n-1}} \rangle (\ell, L_2, \dots, \widehat{L_m}, \dots, L_n) \ell \, d\ell \\
&\quad + \frac{1}{2} \int_{\mathbb{R}_+^2} C^f(L_1, \ell, \ell') \Big(\langle F_{\Sigma_{g-1,n+1}} \rangle (\ell, \ell', L_2, \dots, L_n) \\
&\quad + \sum_{\substack{g_1+g_2=g \\ J_1 \sqcup J_2 = \{L_2,\dots,L_n\}}} \langle F_{\Sigma_{g_1,1+|J_1|}} \rangle (\ell, \ell_{J_1}) \langle F_{\Sigma_{g_2,1+|J_2|}} \rangle (\ell', \ell_{J_2}) \Big) \, \ell \, \ell' \; d\ell \, d\ell'
\end{aligned}$$

and

$$\langle F_P \rangle (L_1, L_2, L_3) = 1, \quad \langle F_T \rangle (L) = \frac{\pi^2}{6} + \frac{L^2}{24} + \frac{1}{2} \int_{\mathbb{R}_+} f(\ell) \ell \, d\ell.$$

Consider the topological recursion of Chekhov, Eynard and Orantin which given a spectral curve

$$(x : \mathfrak{X} \to \mathfrak{X}_0, \omega_{0,1}, \omega_{0,2})$$

produces

- $\omega_{g,n}$ index by $g \geq 0$ and $n \geq 1$, which are denoted the TR amplitudes.

Theorem (Andersen, Borot and Orantin)

Let $(x : \mathfrak{X} \to \mathfrak{X}_0, \omega_{0,1}, \omega_{0,2})$ be a spectral curve and $\omega_{g,n}$ the TR amplitudes.

Let $\mathfrak{r}$ be the set of ramifications points of x. For $r \in \mathfrak{r}$, we introduce local coordinates near $r \in \mathfrak{X}$ and $x(r) \in \mathfrak{X}_0$ such that $x(z) = z^2/2 + c_r$. Let V be the free $\mathbb{C}$-vector space on the set $\mathfrak{r}$.

There exists a family of admissible initial data, parametrized by $\beta \in \mathbb{R}_+$, for the geometric recursion valued in $\mathbf{E}(\Sigma) = \mathscr{C}^0(\mathcal{T}_\Sigma, V^{\otimes \pi_0(\partial\Sigma)})$ for which the GR amplitudes Ω^β_Σ are integrable on $\mathcal{M}_\Sigma(L)$ with respect to the $\nu_{\Sigma,L}$ for any $L \in \mathbb{R}_+$, and with the property that

$$Res_{z'_1 \to r_1} \cdots Res_{z'_n \to r_n} \frac{\omega_{g,n}(z'_1, \ldots, z'_n)}{\prod_{i=1}^n (z_i - z'_i)\, dz_i} = \lim_{\beta \to \infty} \left(\int_{\mathbb{R}^n_+} \prod_{i=1}^n dL_i\, L_i\, e^{-z_i L_i} \int_{\mathcal{M}_{\Sigma_{g,n},L}} \Omega^\beta_\Sigma\, \nu_{\Sigma,L} \right)$$

Theorem (Andersen, Borot and Orantin)

(A, B, C, D) initial data satisfying the admissibility conditions with constants $M(s, \epsilon)$ independent of $\epsilon > 0$, and let Ω be the corresponding GR amplitudes. Then the restriction of Ω_Σ to $\mathcal{M}_\Sigma(L)$ for fixed $L \in \mathbb{R}_+^{\pi_0(\partial\Sigma)}$ is integrable with respect to $\nu_{\Sigma,L}$. For $\Sigma_{g,n}$ connected with genus g and n boundary components set

$$W_{g,n}(L) := \int_{\mathcal{M}_{\Sigma_{g,n}}(L)} \Omega_{\Sigma_{g,n}} \, \nu_{\Sigma_{g,n},L}$$

These functions satisfies topological recursion: First

$$W_{0,3} = A, \qquad W_{1,1}(L) = \int_{\mathcal{M}_T(L)} \Omega_T \, \nu_{T,L}.$$

For any $2g - 2 + n \geq 2$ and $L \in \mathbb{R}_+^{n-1}$ ($W_{0,1} = 0$ and $W_{0,2} = 0$ by convention)

$$\begin{aligned} W_{g,n}(L_1, L) &= \sum_{m=2}^{n} \int_{\mathbb{R}_+} \ell \, B(L_1, L_m, \ell) W_{g,n-1}(\ell, L \setminus \{L_m\}) d\ell \\ &\quad + \frac{1}{2} \int_{\mathbb{R}_+^2} \ell \, \ell' \, C(L_1, \ell, \ell') \Big(W_{g-1,n+1}(\ell, \ell', L) + \\ &\qquad \sum_{\substack{h+h'=g \\ J_1 \dot\cup J_2 = L}} W_{h,1+|J|}(\ell, J) W_{h',1+|J'|}(\ell', J') \Big) d\ell d\ell' \end{aligned}$$

Recall

- $\widetilde{\mathcal{T}}_P^p \cong (\mathbb{R}_+ \times U(1))^{\times 3}$ with coordinates $(L_1, \Theta_1, L_2, \Theta_2, L_3, \Theta_3)$
- $\widetilde{\mathcal{T}}_T^p \cong (\mathbb{R}_+ \times U(1)) \times (\mathbb{R}_+ \times \mathbb{R})$ with Frensel-Nielsen coordinates $(L, \Theta, \ell, \varphi)$.

Now consider the target theory $\mathbf{E}(\Sigma) = \Omega^*(\widetilde{\mathcal{T}}_\Sigma^p)$.

Initial data:

$$A_{\mathrm{WP}} = \exp_\wedge \left(\sum_{i=1}^{3} d\Theta_i \wedge dL_i \right)$$

$$B_{\mathrm{WP}} = B_{\mathrm{MM}}(L_1, L_2, L_3) \exp_\wedge \left(\sum_{i=1}^{2} d\Theta_i \wedge dL_i \right) \wedge d\left(\frac{\Theta_3}{L_3}\right)$$

$$C_{\mathrm{WP}} = C_{\mathrm{MM}}(L_1, L_2, L_3) \exp_\wedge (d\Theta_1 \wedge dL_1) \wedge d\left(\frac{\Theta_2}{L_2}\right) \wedge d\left(\frac{\Theta_3}{L_3}\right)$$

$$D_{\mathrm{WP}} = \exp_\wedge (d\Theta \wedge dL + d\varphi \wedge d\ell)$$

Theorem (Andersen, Borot and Orantin)

For any object Σ in $\mathcal{S}$ the Geometric Recursion applied to the initial data $A_{WP}, B_{WP}, C_{WP}, D_{WP}$ for the target theory $\mathbf{E}(\Sigma) = \Omega^*(\widetilde{\mathcal{T}}_\Sigma^p)$ gives

$$\Omega_\Sigma = \exp(\omega_{WP}).$$

This part is joint with Borot, Charbonnier, Delecroix, Giacchetto, Lewański and Wheeler.

- Consider the bundle of quadratic differentials $Q\mathcal{T}_\Sigma$ over $\mathcal{T}_\Sigma$.
- We have the natural norm $|\cdot|$ on $Q\mathcal{T}_\Sigma$ given by

$$|q| = \int_\Sigma |q \wedge \bar{q}|^{1/2}$$

- There are local holonomy coordinates on $Q\mathcal{T}_\Sigma$ which specifies a lattice subbundle in $Q\mathcal{T}_\Sigma$.
- The **Masur–Veech measure** μ_{MV} on $Q\mathcal{T}_\Sigma$ is defined from this structure by lattice point counting, normalized such that the co-volume of the lattice is one.
- For Y a **measurable subset of the unit norm quadratic differentials** $Q^1\mathcal{T}_\Sigma$ set

$$\mu^1_{\mathrm{MV}}(Y) = (12g - 12 + 4n)\mu_{\mathrm{MV}}(\widetilde{Y}), \qquad \widetilde{Y} = \{tq | t \in (0, \tfrac{1}{2}) \text{ and } q \in Y\}$$

- This measure is clearly Γ_Σ invariant.
- The **Masur–Veech volume** is by definition the total mass

$$MV_{g,n} = \mu^1_{\mathrm{MV}}(Q^1\mathcal{M}_{g,n}) < \infty.$$

Consider the smooth function $f : \mathbb{R}_+ \to \mathbb{R}$ given by

$$f(l) = \frac{1}{e^l + 1}.$$

Let $A^f_K, B^f_K, C^f_K, D^f_K$ be the f twisted Kontsevich initial data and let $\Omega^{MV}_\Sigma \in C^0(\mathcal{T}_\Sigma)$ be the geometric recursion amplitudes obtained from this initial data.

Then Ω^{MV}_Σ is integrable over $\mathcal{M}_\Sigma(L_1, \ldots, L_n)$ w.r.t. the WP-volume form $\nu_{\Sigma,L}$ and we recall our notation

$$\langle \Omega^{\mathrm{MV}}_\Sigma \rangle (L_1, \ldots, L_n) = \int_{\mathcal{M}_\Sigma(L_1,\ldots,L_n)} \Omega^{\mathrm{MV}}_{g,n} \nu_{\Sigma,L}.$$

Theorem

$\langle \Omega^{\mathrm{MV}}_{g,n} \rangle (L_1, \ldots, L_n) = \langle \Omega^{\mathrm{MV}}_{\Sigma_{g,n}} \rangle (L_1, \ldots, L_n)$ *is a polynomial in the L_i's and*

$$MV_{g,n} = \frac{2^{4g-2+n}(4g-4+n)!}{(6g-7+2n)!} \langle \Omega^{\mathrm{MV}}_{g,n} \rangle (0, \ldots, 0).$$

We call $\langle \Omega^{\mathrm{MV}}_{g,n} \rangle (L_1, \ldots, L_n)$ the Masur-Veech polynomials.

Theorem

$$\langle \Omega^{\mathrm{MV}}_{g,n} \rangle(L_1, \ldots, L_n) = \sum_{\substack{d_1,\ldots,d_n \geq 0 \\ d_1+\cdots+d_n \leq 3g-3+n}} F_{g,n}[d_1, \ldots, d_n] \prod_{j=1}^{n} \frac{L_j^{2d_j}}{(2d_j+1)!}.$$

$$F_{0,1}[d_1] = F_{0,2}[d_1, d_2] = 0, \quad F_{0,3}[d_1, d_2, d_3] = \delta_{d_1,d_2,d_3,0}, \quad F_{1,1}[d] = \delta_{d,0}\,\frac{\zeta(2)}{2} + \delta_{d,1}\,\frac{1}{8}$$

$$\begin{aligned} F_{g,n}[d_1, \ldots, d_n] \quad = \quad & \sum_{m=2}^{n} \sum_{a \geq 0} B^{d_1}_{d_m,a}\, F_{g,n-1}[a, d_2, \ldots, \widehat{d_m}, \ldots, d_n] + \\ & + \frac{1}{2} \sum_{a,b \geq 0} C^{d_1}_{a,b} \Big(F_{g-1,n+1}[a, b, d_2, \ldots, d_n] \\ + & \sum_{\substack{h+h'=g \\ J \sqcup J' = \{d_2,\ldots,d_n\}}} F_{h,1+|J|}[a, J]\, F_{h',1+|J'|}[b, J'] \Big), \end{aligned}$$

$$\begin{aligned} B^{i}_{j,k} \quad & = \quad (2j+1)\,\delta_{i+j,k+1} + \delta_{i,j,0}\,\zeta(2k+2), \\ C^{i}_{j,k} \quad & = \quad \delta_{i,j+k+2} + \tfrac{(2j+2a+1)!\,\zeta(2j+2a+2)}{(2j+1)!(2a)!}\,\delta_{i+a,k+1} \\ & \quad + \tfrac{(2k+2a+1)!\,\zeta(2k+2a+2)}{(2k+1)!(2a)!}\,\delta_{i+a,j+1} + \zeta(2j+2)\zeta(2k+2)\delta_{i,0}. \end{aligned}$$

Theorem

For surfaces of genus g with $n > 0$ boundaries, the Masur–Veech volumes are

$$MV_{g,n} = \frac{2^{4g-4+n}(4g-4+n)!}{(6g-7+2n)!} F_{g,n}[0,\ldots,0],$$

while for closed surfaces of genus $g \geq 2$ they are obtained through

$$MV_{g,0} = \frac{2^{4g-2}(4g-4)!}{(6g-6)!} F_{g,1}[1].$$

The following is my own view on and preliminary results concerning the future perspectives of geometric recursion.

Recall B. Zwiebach's formulation of closed String Field Theory.
Part of this theory is the **vertex Hilbert space** V of the theory with its inner product $\langle\cdot,\cdot\rangle$.
The theory provides **brackets** for all $g \geq 0, n \geq 0$ and any sufficiently small $\epsilon \in \mathbb{R}_+$

$$[\cdot,\ldots,\cdot]^{\epsilon}_{g,n} : V^{\times n} \to V,$$

which satisfies the **quantum master equation (QME)** in SFT.
These brackets are determined by the associated **multi-pairings**

$$\{\cdot,\ldots,\cdot\}^{\epsilon}_{g,n} : V^{\times n} \to \mathbb{C}$$

by the formula

$$\{v_1,\ldots,v_n\}^{\epsilon}_{g,n} = \langle v_1, [v_2,\ldots,v_n]^{\epsilon}_{g,n-1}\rangle$$

$v_1,\ldots,v_n \in V$.
These multi-pairings are given by the following expression (for certain top forms $\omega_{g,n}(v_1,\ldots,v_n)$)

$$\{v_1,\ldots,v_n\}^{\epsilon}_{g,n} = \int_{V^{\epsilon}_{g,n}} \omega_{g,n}(v_1,\ldots,v_n),$$

where $V^{\epsilon}_{g,n}$ is a **certain subset** of $M_{g,n}$ which should satisfy the following version of the QME:

$$\partial V^{\epsilon}_{g,n} \cong \left(\bigsqcup_{g_1+g_2=g, n_1+n_2=n+2, n_i \geq 1} V^{\epsilon}_{g_1,n_1} \times V^{\epsilon}_{g_2,n_2} \right) \bigsqcup V^{\epsilon}_{g-1,n+2}$$

We consider the follow function $f_{t,\epsilon} : \mathbb{R}_+ \to \mathbb{R}$ given by

$$f_{t,\epsilon}(l) = \begin{cases} t & l \in [0,\epsilon) \\ 0 & l \in [\epsilon,\infty) \end{cases}$$

We will further require that $\epsilon < \mathrm{argsinh}(1)$.
We now consider the following initial data $A_{MM}^{f_{t,\epsilon}}, B_{MM}^{f_{t,\epsilon}}, C_{MM}^{f_{t,\epsilon}}, D_{MM}^{f_{t,\epsilon}}$ and let

$$\Omega_\Sigma^{\epsilon,t} \in \mathcal{M}(\mathcal{T}_\Sigma)$$

be the result of the geometric recursion applied to this initial data.
For each $\sigma \in \mathcal{T}_\Sigma$ we denote by $n_\epsilon(\sigma)$ the number of simple closed geodesics of length shorter than ϵ.

Theorem

For all $\sigma \in \mathcal{T}_\Sigma$ we have that

$$\Omega_\Sigma^{\epsilon,t}(\sigma) = (1+t)^{n_\epsilon(\sigma)}.$$

Thus, if we let $\Omega_{g,n}^\epsilon = \Omega_{g,n}^{\epsilon,-1}$, we get that

$$\Omega_{g,n}^\epsilon \text{ is the indicator function for the subset } \tilde{V}_{g,n}^\epsilon.$$

where

$$\tilde{V}_{g,n}^\epsilon = \{[\sigma] \in \mathcal{M}_{g,n} \mid \text{ all simple interior closed geodesics on } [\sigma] \text{ have length at least } \epsilon\}$$

Let

$$V^{\epsilon}_{g,n} = \tilde{V}^{\epsilon}_{g,n} \cap \mathcal{M}_{g,n}(\epsilon, \dots, \epsilon).$$

Theorem

The subsets $V^{\epsilon}_{g,n}$ satisfies the quantum master equation

$$\partial V^{\epsilon}_{g,n} \cong \left(\bigsqcup_{g_1+g_2=g, n_1+n_2=n+2, n_i \geq 1} V^{\epsilon}_{g_1,n_1} \times V^{\epsilon}_{g_2,n_2} \right) \bigsqcup V^{\epsilon}_{g-1,n+2}$$

Since $\Omega^{\epsilon}_{g,n}$ is the indicator function of $\tilde{V}^{\epsilon}_{g,n}$ we of course have that

$$\int_{V^{\epsilon}_{g,n}} \omega_{g,n}(v_1, \dots, v_n) = \int_{\mathcal{M}_{g,n}(\epsilon,\dots,\epsilon)} \Omega^{\epsilon}_{g,n} \omega_{g,n}(v_1, \dots, v_n).$$

Let

$$V^{\epsilon}_{g,n} = \tilde{V}^{\epsilon}_{g,n} \cap \mathcal{M}_{g,n}(\epsilon, \ldots, \epsilon).$$

Theorem

The subsets $V^{\epsilon}_{g,n}$ satisfies the quantum master equation

$$\partial V^{\epsilon}_{g,n} \cong \left(\coprod_{g_1+g_2=g, n_1+n_2=n+2, n_i \geq 1} V^{\epsilon}_{g_1,n_1} \times V^{\epsilon}_{g_2,n_2} \right) \coprod V^{\epsilon}_{g-1,n+2}$$

Since $\Omega^{\epsilon}_{g,n}$ is the indicator function of $\tilde{V}^{\epsilon}_{g,n}$ we of course have that

$$\int_{V^{\epsilon}_{g,n}} \omega_{g,n}(v_1, \ldots, v_n) = \int_{\mathcal{M}_{g,n}(\epsilon,\ldots,\epsilon)} \Omega^{\epsilon}_{g,n} \omega_{g,n}(v_1, \ldots, v_n).$$

It is likely that we can further build $\Omega^{\epsilon}_{g,n}\omega_{g,n}(v_1, \ldots, v_n)$ via **geometric recursion** (since $\omega_{g,n}(v_1, \ldots, v_n)$ is build from the usual conformal field theory constructions which satisfies factorization) and then we will get that that the string brackets

$$\{v_1, \ldots, v_n\}^{\epsilon}_{g,n} = \int_{V^{\epsilon}_{g,n}} \omega_{g,n}(v_1, \ldots, v_n)$$

can be computed by **topological recursion**!

We are currently working with other candidate target theories:

Functions on Hitchin's higher Teichmüller components:
One considers Hitchin's component of the $\mathrm{SL}(n, \mathbb{R})$ moduli space and then normalized logarithms of spectral radius holonomy functions in place of length functions, precisely as done by M. Bridgeman, R. Canary, F. Labouri & A. Sambarino when they construct the Pressure Metric on this component.

Ex. 3. Functions on Teichmüller space via spectral theory:

$$\mathbf{E}(\Sigma) = \mathcal{C}^0(\mathcal{T}_\Sigma), \qquad \Omega_\Sigma(\sigma) = \mathrm{Tr}(f(-\Delta_\sigma))$$

• $f : \mathbb{R} \to \mathbb{C}$ is sufficiently fast decaying at infinity and Δ_σ Dirichlet-Laplace-Beltrami operator on the Riemann surface Σ_σ, $\sigma \in \mathcal{T}_\Sigma$.
The Selberg trace formula expresses Ω_Σ as sum over geodesics on Σ:

$$\begin{aligned}
\mathrm{Tr}(f(-\Delta_\sigma)) &= \frac{2g+n-1}{2}\int_{\mathbb{R}} \tilde{f}(p)p\tanh(\pi p)dp \\
&+ \sum_{\gamma\in G_p}\sum_{k=1}^{\infty} \frac{\ell_\sigma(\gamma)}{4\sinh(k\frac{\ell_\sigma(\gamma)}{2})} g(k\ell_\sigma(\gamma)) \\
&- \sum_{\gamma\in G'_p}\sum_{k=1}^{\infty} \frac{\ell_\sigma(\gamma)}{4\cosh((k+\frac{1}{2})\frac{\ell_\sigma(\gamma)}{2})} g((k+\frac{1}{2})\ell_\sigma(\gamma)) \\
&- \sum_{i=1}^{n}\sum_{k=1}^{\infty} \frac{L_i}{4\cosh(k\frac{L_i}{2})} g(kL_i) \\
&- \frac{L}{4}g(0),
\end{aligned}$$

where G_P and G'_P are certain sets of primitive geodesics on Σ,

$$g(y) = \int_{\mathbb{R}} \tilde{f}(x)e^{ixy}dx$$

and $f(\lambda) = \tilde{f}(p)$, $\lambda = p^2 + \frac{1}{4}$.

However, if one instead consider another category $\mathcal{S}'$ of surfaces:

Objects: Compact oriented surfaces with corners with a marked point on each boundary (which must be a corner, if the component has corners and we set $c(\Sigma)$ in total number of corners and marked points) on the boundary Σ with $\chi(\Sigma) - c(\Sigma) < -1$ together with an orientation of the boundary, such that $\partial\Sigma = \partial_-\Sigma \cup \partial_+\Sigma$, and such that the inclusion map $\partial_-\Sigma \subset \Sigma$ induces $\pi_0(\partial_-\Sigma) \cong \pi_0(\Sigma)$.

Morphisms: Isotopy classes of orientation preserving diffeomorphisms which preserves marked points and orientations on the boundary modulo isotopies which also preserves all this structure.

However, if one instead consider another category $\mathcal{S}'$ of surfaces:

Objects: Compact oriented surfaces with corners with a marked point on each boundary (which must be a corner, if the component has corners and we set $c(\Sigma)$ in total number of corners and marked points) on the boundary Σ with $\chi(\Sigma) - c(\Sigma) < -1$ together with an orientation of the boundary, such that $\partial\Sigma = \partial_-\Sigma \cup \partial_+\Sigma$, and such that the inclusion map $\partial_-\Sigma \subset \Sigma$ induces $\pi_0(\partial_-\Sigma) \cong \pi_0(\Sigma)$.

Morphisms: Isotopy classes of orientation preserving diffeomorphisms which preserves marked points and orientations on the boundary modulo isotopies which also preserves all this structure.

Suppose now we have a functor

$$\mathbf{E} : \mathcal{S} \to \mathcal{C}.$$

The recursion now proceeds by iteratively removing embedded **triangles** from Σ.

A scheme similar to the one presented for Geometric Recursion in this talk also works in this case and one in facts gets what we call **Open Geometric Recursion**.

This allows us to get recursion for the spectral functions $\Omega_\Sigma(\sigma) = \mathrm{Tr}(f(-\Delta_\sigma))$ via the Selberg trace formula and in fact also get :

A recursion in (g, n, c) for their expectation values: $\langle \Omega_{\Sigma_{g,n,c}} \rangle$.

Answers a long standing open problem in spectral theory with application in string theory.

The true category $\mathcal{C}$:

Objects: An object $\mathbf{V}$ of $\mathcal{C}$ is a directed set $\mathscr{I}$ and an inverse system over $\mathscr{I}$ of objects

$$(V^{(i)}, (|\cdot|_{\alpha}^{(i)})_{\alpha \in \mathscr{A}^{(i)}})_{i \in \mathscr{I}}$$

of $\mathcal{V}$. Inside the projective limit V of the $(V^{(i)})_{i \in \mathscr{I}}$ we have the important subspace $V' := \{v \in V \quad | \quad \forall i \in \mathscr{I}, \; ||v||^{(i)} < +\infty\} \subset V$, where $\|v\|^{(i)} := \sup_{\alpha \in \mathscr{A}^{(i)}} |v|_{\alpha}^{(i)}$.

Morphisms: A morphism Φ of $\mathcal{C}$ from an object $\mathbf{V}_1$ to another $\mathbf{V}_2$, is an inverse system of continuous linear maps

$$\Phi^{i,j} : V_1^{(i)} \to V_2^{(j)}, \qquad i \in \mathscr{I}_1, \quad j \leq h(i)$$

over an order preserving map $h : \mathscr{I}_1 \to \mathscr{I}_2$, such that the induced continuous linear map $\Phi \; : \; V_1 \to V_2$ satisfies $\Phi(V_1') \subseteq V_2'$.

Recall S_Σ is the set of multi-curves in Σ.

Definition

A ($\mathcal{C}$-valued) target theory is a functor $\mathbf{E}$ from $\mathcal{S}$ to the category $\mathcal{C}$, such that morphisms in $\mathcal{S}$ are send to isometries in $\mathcal{C}$, together with the following extra structure. For each object Σ of $\mathcal{S}$ with

$$\mathbf{E}(\Sigma) = \left(E^{(i)}(\Sigma), (|\cdot|^{(i)}_\alpha)_{\alpha \in \mathscr{A}^{(i)}_\Sigma}\right)_{i \in \mathscr{I}_\Sigma},$$

we require the functorial data of *lengths functions*

$$l^{(i)}_\alpha \; : \; S_\Sigma \longrightarrow \mathbb{C} \setminus \{0\}$$

indexed by $i \in \mathscr{I}_\Sigma$ and $\alpha \in \mathscr{A}^{(i)}_\Sigma$. This data must satisfy the following properties.

Recall S_Σ is the set of multi-curves in Σ.

Definition

A ($\mathcal{C}$-valued) target theory is a functor $\mathbf{E}$ from $\mathcal{S}$ to the category $\mathcal{C}$, such that morphisms in $\mathcal{S}$ are send to isometries in $\mathcal{C}$, together with the following extra structure. For each object Σ of $\mathcal{S}$ with

$$\mathbf{E}(\Sigma) = \left(E^{(i)}(\Sigma), (|\cdot|^{(i)}_\alpha)_{\alpha\in\mathscr{A}^{(i)}_\Sigma}\right)_{i\in\mathscr{I}_\Sigma},$$

we require the functorial data of *lengths functions*

$$l^{(i)}_\alpha \; : \; S_\Sigma \longrightarrow \mathbb{C}\setminus\{0\}$$

indexed by $i\in\mathscr{I}_\Sigma$ and $\alpha\in\mathscr{A}^{(i)}_\Sigma$. This data must satisfy the following properties.

Polynomial growth axiom.
For each $i\in\mathscr{I}_\Sigma$, $\alpha\in\mathscr{A}^{(i)}_\Sigma$ and $L\in\mathbb{R}_+$, the set

$$N^{(i)}_\alpha(\Sigma,L) = \{\gamma\in S_\Sigma \mid |l^{(i)}_\alpha(\gamma)|\leq L\}$$

is finite and there exists $m_i(\Sigma), d_i(\Sigma)\in\mathbb{R}_+$, such that

$$\sup_{\alpha\in\mathscr{A}^{(i)}(\Sigma)} |N^{(i)}_\alpha(\Sigma,L)| \leq m_i(\Sigma)L^{d_i(\Sigma)}.$$

LOWER BOUND AXIOM.
For any $i \in \mathscr{I}_\Sigma$, there exists $\epsilon_i > 0$ such that

$$\inf \left\{ |l_\alpha^{(i)}(\gamma)| \mid (\alpha, \gamma) \in \mathscr{A}_\Sigma^{(i)} \times S_\Sigma \right\} \geq \epsilon_i .$$

SMALL PAIR OF PANTS
For any $i \in \mathscr{I}_\Sigma$, there exists $Q_i > 0$, s.t. $\forall \alpha \in \mathscr{A}_\Sigma^i$

$$| \{ P \in \mathcal{P}(\Sigma) \mid l_\alpha^{(i)}(\partial(P \cap \Sigma^\circ)) \leq l_\alpha^{(i)}(\partial\Sigma \cap \partial P) \} | \leq Q_i$$

UNION AXIOM.
For any two objects Σ_1 and Σ_2 of $\mathcal{S}$, we ask for a bilinear morphism

$$\sqcup \ : \ \mathbf{E}(\Sigma_1) \times \mathbf{E}(\Sigma_2) \to \mathbf{E}(\Sigma_1 \cup \Sigma_2),$$

compatible with associativity of cartesian products and associativity of unions.

GLUEING AXIOM.
For any two objects Σ_1 and Σ_2 in $\mathcal{S}$, and a subset $\beta \subset \pi_0(\partial\Sigma_1) \times \pi_0(\partial\Sigma_2)$ consisting of disjoint pairs. We ask for a bilinear morphism

$$\Theta_\beta \ : \ \mathbf{E}(\Sigma_1) \times \mathbf{E}(\Sigma_2) \to \mathbf{E}(\Sigma_1 \cup_\beta \Sigma_2),$$

which is compatible with the glueing of morphisms, with associativity of glueings and with the union morphisms.

Definition

Initial data for a given target theory **E** are assignments

- $A, C \in \mathbf{E}(P)^{\Gamma_P}$.
- $B^b \in \mathbf{E}(P)$ for $b \in \pi_0(\partial_+ P)$ such that $\varphi(B^b) = B^{\varphi(b)}$ for all $\varphi \in \Gamma(P)$.
- $D \in \mathbf{E}(T)^{\Gamma_T}$.

Definition

Initial data for a given target theory $\mathbf{E}$ are assignments

- $A, C \in \mathbf{E}(P)^{\Gamma_P}$.
- $B^b \in \mathbf{E}(P)$ for $b \in \pi_0(\partial_+ P)$ such that $\varphi(B^b) = B^{\varphi(b)}$ for all $\varphi \in \Gamma(P)$.
- $D \in \mathbf{E}(T)^{\Gamma_T}$.

Definition

The initial data is called *admissible* if
• $A \in E'(P)$, $D \in E'(T)$
and

$$\left([x]_+ = \frac{1+\mathrm{Sign(x)}}{2}x\right)$$

Decay axiom. For any connected object Σ in $\mathcal{S}$, any $P \in \mathcal{P}(\Sigma)$, we require that for any $(i,j) \in \mathscr{I}_P \times \mathscr{I}_{\Sigma_c}$ and $k \in \mathscr{I}_\Sigma$ such that $k \leq h_P(i,j)$, any $\alpha \in \mathscr{A}_\Sigma^{(k)}$, there exists $s_k > d_k(\Sigma)$ and functorial $M_{i,j,k}(\Sigma) > 0$ such that

- if P shares two boundary components with Σ, say $\partial_- P$ and b, then $\forall v \in E'(\Sigma_c)^{\Gamma(\Sigma_c)}$

$$\left|\Theta_{b'}^{i,j,k}(B^b, v)\right|_\alpha^{(k)} \leq M_{i,j,k}(\Sigma)\, \|v\|^{(j)}\, (1 + [l_\alpha^{(i)}(\partial P \cap \Sigma^\circ) - l_\alpha^{(i)}(\partial P \cap \partial\Sigma)]_+)^{-s_k}.$$

- if P shares only one boundary component with Σ, then $\forall v \in E'(\Sigma_c)^{\Gamma(\Sigma_c)}$

$$\left|\Theta_{b,b'}^{i,j,k}(C, v)\right|_\alpha^{(k)} \leq M_{i,j,k}(\Sigma)\, \|v\|^{(j)}(1 + [l_\alpha^{(i)}(\partial P \cap \Sigma^\circ) - l_\alpha^{(i)}(\partial P \cap \partial\Sigma)]_+)^{-s_k}.$$

The **decay axiom:** $\forall s > 0$, any $(i,j) \in \mathscr{I}_{\mathrm{P}} \times \mathscr{I}_{\Sigma_c}$ and $k \in \mathscr{I}_{\Sigma}$, any $\alpha \in \mathscr{A}_{\Sigma}^{(k)}$ that

$$\sum_{P \in \mathcal{P}_B(\Sigma)} \left|\Theta_{b'}^{i,j,k}(B^b, \Omega_{\Sigma_c})\right|_{\alpha}^{(k)} \leq M_{i,j,k}(\Sigma) \|\Omega_{\Sigma_c}\|^{(j)} \, \zeta_{\alpha}(s)\,,$$

where

$$\zeta_{\alpha}^{(i)}(s) = \sum_{P \in \mathcal{P}_B(\Sigma)} \left(1 + [l_{\alpha}^{(i)}(\partial P \cap \Sigma^{\circ}) - l_{\alpha}^{(i)}(\partial P \cap \partial \Sigma)]_{+}\right)^{-s} \in (0, +\infty]\,.$$

The **polynomial growth axiom + small pair of pants:** There exist $s_k > d_k(\Sigma)$ such that $\zeta_{\alpha}(s_k)$ is finite.

The **lower bound axiom + small pair of pants:** There exists a finite constant M'_k such that

$$\sup_{\alpha \in \mathscr{A}_{\Sigma}^{(k)}} \zeta_{\alpha}^{(i)}(s_k) \leq M'_k\,.$$

Thus we get that

$$\sum_{P \in \mathcal{P}_B(\Sigma)} \left|\Theta_{b'}^{i,j,k}(B^b, \Omega_{\Sigma_c})\right|_{\alpha}^{(k)} \leq M_{i,j,k}(\Sigma) \, \|\Omega_{\Sigma_c}\|^{(j)} \, M'_k,$$

e.g. the series $\sum_{P \in \mathcal{P}_B(\Sigma)} \Theta_{b'}^{i,j,k}(B^b, \Omega_{\Sigma_c})$ is absolutely convergent in $E^{(k)}(\Sigma)$.

Let $K_\Sigma(\varepsilon) := \{\sigma \in \widetilde{\mathcal{T}}^p_\Sigma \mid \mathrm{sys}_\sigma \geq \varepsilon\}$ and $E^\varepsilon(\Sigma) := \mathcal{C}^0(K_\Sigma(\varepsilon))$.

We have a family of seminorms indexed by the set $\mathscr{A}^\varepsilon_\Sigma$ of compact subsets of $K_\Sigma(\varepsilon)$, which makes it a locally convex, Hausdorff, complete topological vector spaces, and we have continuous restriction maps $E^\varepsilon(\Sigma) \to E^{\varepsilon'}(\Sigma)$ whenever $\varepsilon \leq \varepsilon'$.

One then easily checks that $E(\Sigma) := \mathcal{C}^0(\widetilde{\mathcal{T}}^p_\Sigma)$ is the projective limit of these spaces over the directed set $\mathbb{R}_+$.
We have seminorms

$$\|f\|_\varepsilon = \sup_{\sigma \in K_\Sigma(\varepsilon)} |f(\sigma)|$$

and a subspace

$$E'(\Sigma) = \{f \in \mathcal{C}^0(\widetilde{\mathcal{T}}_\Sigma) \mid \forall \varepsilon > 0, \quad \|f\|_\varepsilon < +\infty\}.$$

For any $\varepsilon > 0$ and K a compact subset of $K_\Sigma(\varepsilon)$, we use the hyperbolic length ℓ_σ to define the length functions,

$$\forall \gamma \in S_\Sigma, \qquad l_K^{(\epsilon)}(\gamma) = \min_{\sigma \in K} \ell_\sigma(\gamma).$$

Since K is compact for any $\sigma \in K$, there exists a constant $c_K \in (0,1)$ such that

$$c_K\, \ell_\sigma(\gamma) \leq l_K^{(\epsilon)}(\gamma).$$

As the systole is bounded below by construction on each $K_\Sigma(\varepsilon)$, we deduce that the length functions satisfy the Lower bound axiom.

A result of Rivin (refined by Mirzakhani) guarantees that the number of $\gamma \in S_\Sigma$ with $l_K^{(\epsilon)}(\gamma) \leq L$ grows slower than a power of L, thus we get the Polynomial growth axiom.
Work of Hugo Parlier provides the Small pair of pants axiom.

Congratulations with your creations of the

IMSA!